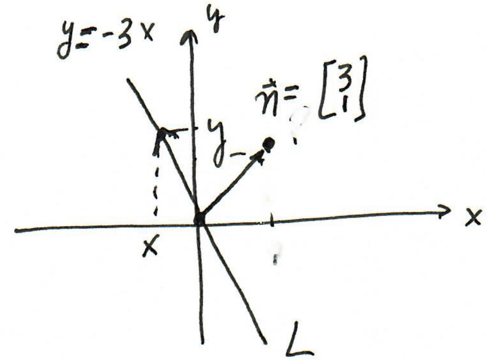
1.3 LINES AND PLANES.

In $\mathbb{R}^2$

Consider the line

$y = -3x$

or $3x + y = 0$



Another representation

$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$, Since $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3x + y$

By calling $\vec{n} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ with head in L

We notice that $\vec{n}$ is orthogonal to any vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, where the point $P(x,y)$ is in L.

Then, we have two representations of this line:

a) $\boxed{3x + y = 0}$ General form.

b) $\boxed{\vec{n} \cdot \vec{x} = 0,}$ $\vec{n} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ normal form.

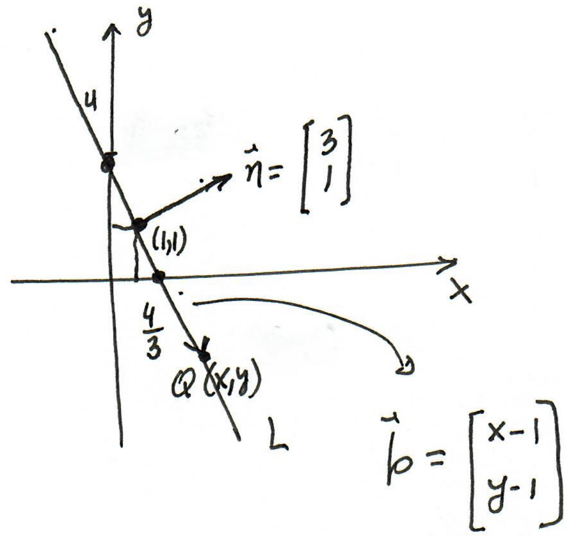
Consider now the line

General form

$$L: y = -3x + 4 \quad \text{or} \quad \boxed{3x + y = 4} \qquad (2.1)$$

If $x = 1$

$y = -3(1) + 4 = 1$, so line pass thru $P(1,1)$.



Then (from graph is clear) that if $Q^{(x,y)}$ is in L

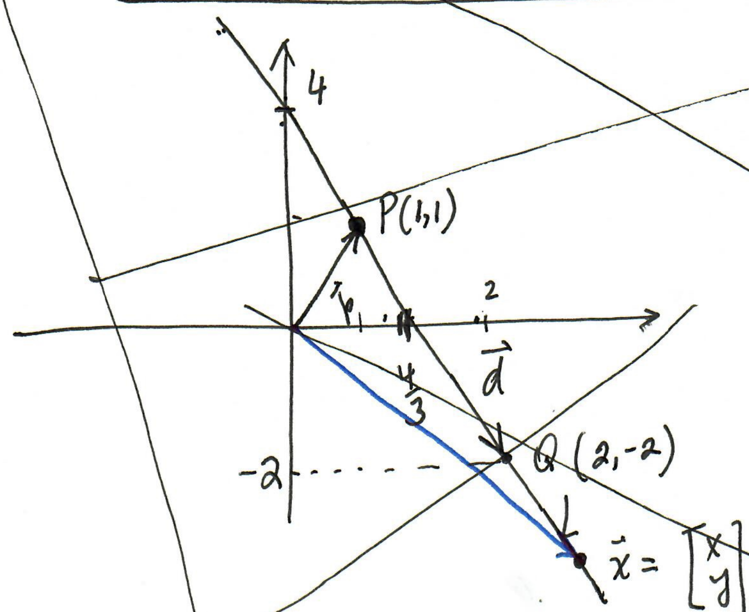
$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = 0$$

or $\boxed{\vec{n} \cdot (\vec{x} - \vec{p}) = 0}$, (2.2) or $3(x-1) + (1)(y-1) = 0$

or $3x + y = 3 + 1 = 4$

normal form

Another representation (Vector form)



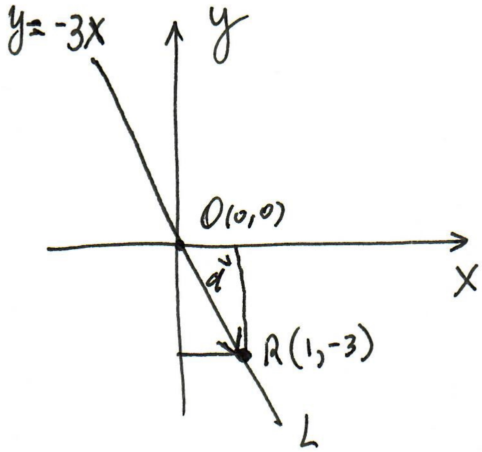
$$\vec{d} = \vec{PQ} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

The points on L $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be obtained by the sum of two vectors

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Another Representation (Vector form)

Back to the Line $y=-3x$ or $3x+y=0$



if $x=1$, then $y=-3$

$R(1,-3)$ a point in L

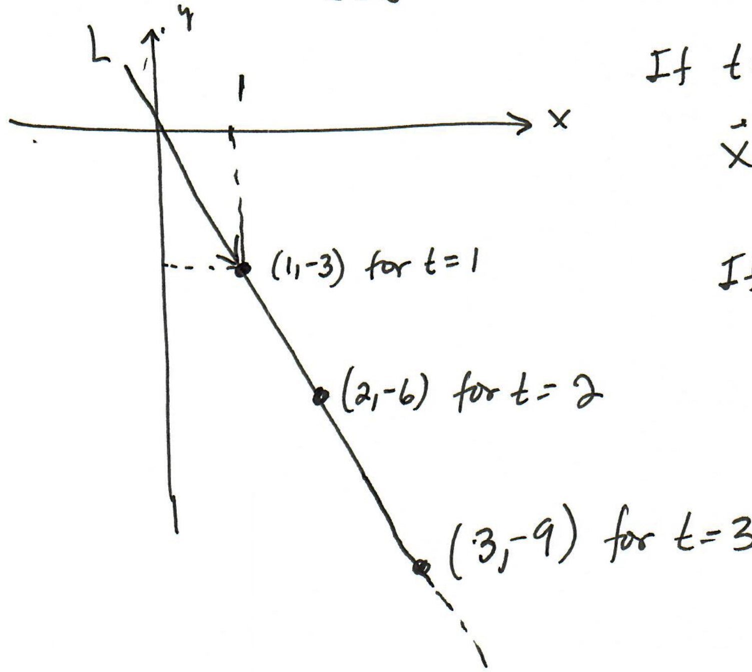
$\vec{d}=\begin{bmatrix}1\\-3\end{bmatrix}$ is a direction vector for L

and L can be written as

$$\vec{x}=t\,\vec{d}, \quad \text{where } \vec{d}=\begin{bmatrix}1\\-3\end{bmatrix} \text{ and } t\in\mathbb{R}.$$

meaning of "t"

If $t=1$ $\quad \vec{x}=\begin{bmatrix}x\\y\end{bmatrix}=t\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}1\\-3\end{bmatrix}$ $\qquad (3.1)$



If $t=2$,

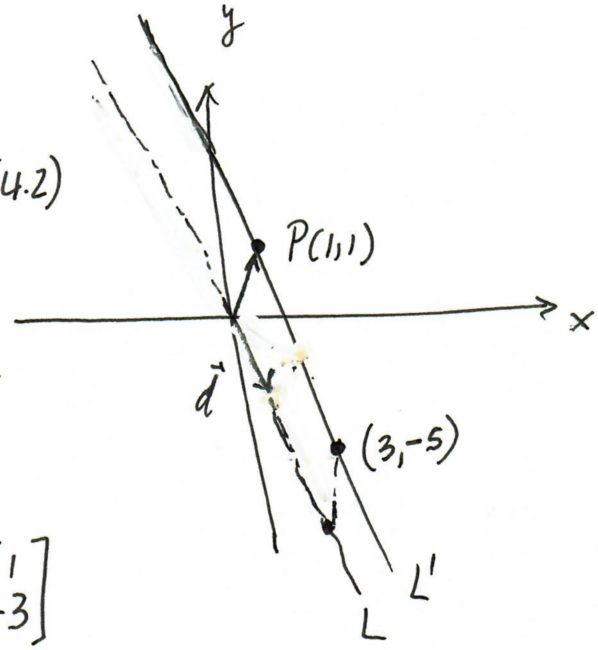
$\vec{x}=\begin{bmatrix}x\\y\end{bmatrix}=2\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}2\\-6\end{bmatrix}$

If $t=3$

$\begin{bmatrix}x\\y\end{bmatrix}=3\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}3\\-9\end{bmatrix}$

Equ. (3.1) can be written as

$\boxed{\vec{x} = t\,\vec{d}}$ Vector form. (4.1)



Now, consider the line

$\boxed{3x+y=4}$ (4.2)

Clearly, L' is parallel to L.

So they have the same direction vector, $\vec{d} = \begin{bmatrix}1\\-3\end{bmatrix}$

Notice that any point $\begin{bmatrix}x\\y\end{bmatrix} \in L'$

can be obtained as $\begin{bmatrix}x\\y\end{bmatrix} = t\begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$

For example, the point Q(3,-5) is in L' (satisfies (4.2))

and $\begin{bmatrix}3\\-5\end{bmatrix} = \overset{=2}{t}\begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = 2\begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\-5\end{bmatrix}$

If $t=3$ $3\begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\-8\end{bmatrix}$ which also satisfies (4.2)

$3x+y = 3(4)+(-8) = 4$ ✓

Basically, what we are doing is translating the line L to L' adding the vector $\begin{bmatrix}1\\1\end{bmatrix} = \vec{p}$

to any vector $t\begin{bmatrix}1\\-3\end{bmatrix}$ lying in L.

So,

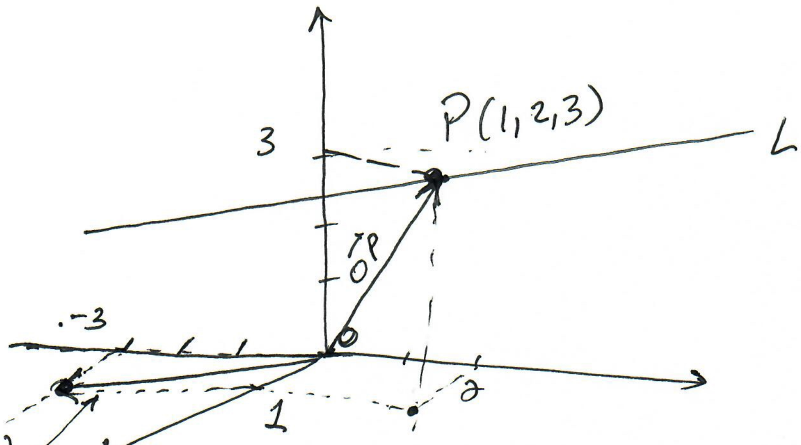
$\begin{bmatrix}x\\y\end{bmatrix}$ is in L', if $\begin{bmatrix}x\\y\end{bmatrix} = t\begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$ or

$\boxed{\vec{x} = t\vec{d} + \vec{p},}$ $\quad \vec{d} = \begin{bmatrix}1\\-3\end{bmatrix}, \ \vec{p} = \begin{bmatrix}1\\1\end{bmatrix}$

<u>Vector form</u>

1.3 Lines in $\mathbb{R}^3$

Construct the line L passing through point $P(1,2,3)$ with direction vector $\vec{d}=\begin{bmatrix}1\\-3\\0\end{bmatrix}$.



$\vec{d}=\begin{bmatrix}1\\-3\\0\end{bmatrix}$

$\vec{x}$ is in L, $\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}1\\-3\\0\end{bmatrix}+\begin{bmatrix}1\\2\\3\end{bmatrix}$

or $\vec{x}=t\vec{d}+\vec{OP}$.

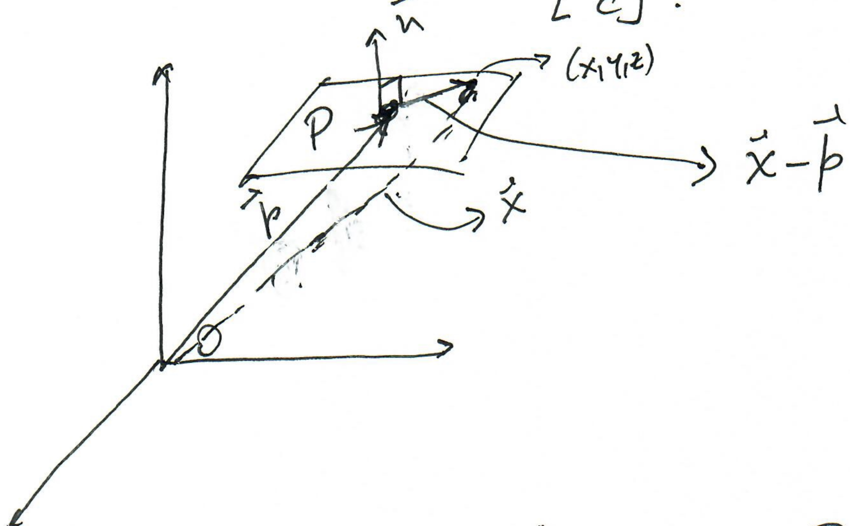
Therefore, to derive the equation for a line L in $\mathbb{R}^2$ or $\mathbb{R}^3$, we need a direction vector $\vec{d}$ an a point on the line $P(x_0,y_0,z_0)$

$$\vec{x}=t\vec{d}+\vec{p}$$

where $\vec{p}=\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix}$.

Equations of a Plane in $\mathbb{R}^3$

A plane $\mathcal{P}$ equation can be obtained by knowing a point $P(x_0, y_0, z_0)$ in $\mathcal{P}$ and a normal vector to $\mathcal{P}$ $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.



In fact, any other point $\vec{x} = (x, y, z)$ on $\mathcal{P}$ satisfies the vector equation:

$$(\vec{x} - \vec{p}) \cdot \vec{n} = 0 \qquad \underline{\text{Vector form}}$$

or

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$\boxed{ax + by + cz = ax_0 + by_0 + cz_0 = d}$$

<u>General form</u>

<u>Example:</u> Find the equation of the plane that passes through $P(1,2,4)$ and has normal $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

or

$$\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-2 \\ z-4 \end{bmatrix} = 0$$

$$3x - y + 5z = 3 - 2 + 20 = 21$$

or

$$\boxed{3x - y + 5z = 21}$$